

The Wigner-Yanase Entropy is not Subadditive

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Wigner and Yanase introduced in 1963 the Wigner-Yanase entropy defined as minus the skew information of a state with respect to a conserved observable. They proved that the Wigner-Yanase entropy is a concave function in the state and conjectured that it is subadditive with respect to the aggregation of possibly interacting subsystems. While this turned out to be true for the quantum-mechanical entropy, we negate the conjecture for the Wigner-Yanase entropy by providing a counter example.

KEY WORDS: Wigner-Yanase entropy; subadditivity.

1. INTRODUCTION

The Wigner-Yanase-Dyson entropy $S_p(\rho, k)$ is defined⁽¹⁰⁾ by setting

$$S_p(\rho, k) = \frac{1}{2} \text{Tr}[\rho^p, k][\rho^{1-p}, k] \quad 0 < p < 1,$$

where ρ is a state (or density matrix) and k is a conserved (self-adjoint) observable. Since $S_p(\rho, k)$ is homogeneous in ρ we may regard it as a function defined on the set of all positive definite operators on a Hilbert space H . If the dimension of H is infinite, we usually impose the condition that k is a trace class operator. Note also that $S_p(\rho, k)$ is minus the Wigner-Yanase-Dyson skew information in the state ρ with respect to the (conserved) observable k .

Wigner and Yanase proved concavity in the state ρ of the Wigner-Yanase entropy

$$S(\rho, k) = \frac{1}{2} \text{Tr}[\rho^{1/2}, k]^2$$

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and conjectured that also the Wigner-Yanase-Dyson entropy is concave in ρ for $0 < p < 1$. The conjecture was eventually proved by Lieb.⁽⁵⁾ The result is in line with the well known concavity of the quantum-mechanical entropy

$$S(\rho) = -\text{Tr } \rho \log \rho.$$

Wigner and Yanase also conjectured⁽¹⁰⁾ that the Wigner-Yanase entropy is subadditive in the following sense. Let $H_{12} = H_1 \otimes H_2$ be a tensor product of two Hilbert spaces H_1 and H_2 , and let ρ_{12} be a positive definite operator on H_{12} . In physical applications one also requires that $\text{Tr} \rho_{12} = 1$. The partial trace $\rho_1 = \text{Tr}_1 \rho_{12}$ is the operator on H_1 defined by setting

$$(\xi | \rho_1 \eta) = \sum_{i \in I} (\xi \otimes e_i | \rho_{12}(\eta \otimes e_i)) \quad \xi, \eta \in H_1$$

where $(e_i)_{i \in I}$ is any orthonormal basis in H_2 . We similarly define the partial trace $\rho_2 = \text{Tr}_2 \rho_{12}$ on H_2 . Let k_1 (respectively k_2) be a self-adjoint operator on H_1 (respectively H_2) and define

$$k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2.$$

Subadditivity of the Wigner-Yanase-Dyson entropy is the condition

$$S_p(\rho_{12}, k_{12}) \leq S_p(\rho_1, k_1) + S_p(\rho_2, k_2). \quad (1)$$

Wigner and Yanase proved subadditivity (1) if $\rho_{12} = \rho_1 \otimes \rho_2$ is a simple tensor of density matrices ρ_1 and ρ_2 (with equality), or if ρ_{12} is a pure state.⁽¹⁰⁾ Lieb also noted⁽⁵⁾ that (1) holds if $k_1 = 0$ or $k_2 = 0$.

The notion of subadditivity for the classical entropy with respect to the aggregation of not necessarily isolated systems goes back to Gibbs⁽²⁾ and were later used by Kolmogorov and Sinai. A historical account may be found in Werhl,⁽⁹⁾ cf. also Ruelle⁽⁸⁾ Proposition 7.2.6). It was therefore natural to expect the same properties of the quantum-mechanical entropy. Subadditivity of the quantum-mechanical entropy was proved by Landford and Robinson⁽⁴⁾ who cited earlier partial results by Delbrück and Molière, and R. Jost, cf. also Araki and Lieb,⁽¹⁾ while strong subadditivity were conjectured by Landford and Robinson⁽⁴⁾ and proved by Lieb and Ruskai,⁽⁶⁾ cf. also.⁽⁷⁾

The notion of strong subadditivity for the Wigner-Yanase-Dyson entropy is defined in the following way. Let $H_{123} = H_1 \otimes H_2 \otimes H_3$ be a tensor product of three Hilbert spaces H_1 , H_2 and H_3 and let ρ_{123} be a positive definite operator on H_{123} . We consider the partial traces

$$\rho_2 = \text{Tr}_2 \rho_{123}, \quad \rho_{12} = \text{Tr}_{12} \rho_{123}, \quad \rho_{23} = \text{Tr}_{23} \rho_{123}$$

and define

$$k_{123} = k_1 \otimes 1_2 \otimes 1_3 + 1_1 \otimes k_2 \otimes 1_3 + 1_1 \otimes 1_2 \otimes k_3,$$

where k_1, k_2 and k_3 are self-adjoint operators on H_1, H_2 and H_3 respectively. Strong subadditivity of the Wigner-Yanase-Dyson entropy is the condition

$$S_p(\rho_{123}, k_{123}) + S_p(\rho_2, k_2) \leq S_p(\rho_{12}, k_{12}) + S_p(\rho_{23}, k_{23}), \tag{2}$$

where $k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2$ and $k_{23} = k_2 \otimes 1_3 + 1_2 \otimes k_3$.

Strong subadditivity (SSA) is for the Wigner-Yanase-Dyson entropy a stronger condition than just subadditivity (SA). This may be inferred² in the following way. We set $H_2 = \mathbf{C}$ and let

$$\rho_{13} = \sum_i x_i \otimes z_i, \quad x_i \in B(H_1), z_i \in B(H_3)$$

be a positive semi-definite operator on the tensor product $H_1 \otimes H_3$ and set

$$\rho_{123} = \sum_i x_i \otimes 1_2 \otimes z_i.$$

We calculate the partial trace

$$\text{Tr}_{13} \rho_{123} = \rho_{13},$$

and by setting $k_2 = 1_2$, we obtain $S_p(\rho_{123}, k_{123}) = S_p(\rho_{13}, k_{13})$ and

$$S_p(\rho_2, k_2) = 0, \quad S_p(\rho_{12}, k_{12}) = S_p(\rho_1, k_1), \quad S_p(\rho_{23}, k_{23}) = S_p(\rho_3, k_3).$$

Strong subadditivity thus entails

$$S_p(\rho_{13}, k_{13}) \leq S_p(\rho_1, k_1) + S_p(\rho_3, k_3),$$

which is subadditivity.

2. THE WY-ENTROPY IS NOT SUBADDITIVE

We shall demonstrate that the Wigner-Yanase entropy is not subadditive (and therefore not strongly subadditive either).

Consider the tensor product $H_{12} = H_1 \otimes H_2$ of two Hilbert spaces, each of dimension 2, with fixed orthonormal bases (e_1, e_2) in H_1 and (f_1, f_2) in H_2 . We introduce the positive definite operator ρ_{12} on H_{12} with matrix representation

$$\rho_{12} = \begin{pmatrix} 7 & 5 & 5 & 6 \\ 5 & 6 & 2 & 5 \\ 5 & 2 & 6 & 5 \\ 6 & 5 & 5 & 7 \end{pmatrix}$$

and eigenvalues $\frac{1}{2}(21 + 5\sqrt{17})$, 4, 1, $\frac{1}{2}(21 - 5\sqrt{17})$. The coordinates of for example $\rho_{12}(e_2 \otimes f_1)$ are then given by column no. 3 in the matrix representing ρ_{12} .

²The author is indebted to E. Lieb and R. Seiringer for providing this argument.

The positive definite square root $\rho_{12}^{1/2}$ is represented by

$$\rho_{12}^{1/2} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

and have eigenvalues $\frac{1}{2}(5 + \sqrt{17})$, 2, 1, $\frac{1}{2}(5 - \sqrt{17})$.

We also consider the operators on H_1 and H_2 given by

$$k_1 = \begin{pmatrix} 10 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$k_2 = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix},$$

and calculate

$$k_1 \otimes 1_2 = \begin{pmatrix} 10 & 0 & 1 & 0 \\ 0 & 10 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad 1_1 \otimes k_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 10 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 10 \end{pmatrix}$$

and

$$k_{12} = k_1 \otimes 1_2 + 1_1 \otimes k_2 = \begin{pmatrix} 11 & 1 & 1 & 0 \\ 1 & 20 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 11 \end{pmatrix}.$$

The commutator

$$[\rho_{12}^{1/2}, k_{12}] = \begin{pmatrix} 0 & 10 & -8 & 0 \\ -10 & 0 & 0 & -10 \\ 8 & 0 & 0 & 8 \\ 0 & 10 & -8 & 0 \end{pmatrix}$$

has a square

$$[\rho_{12}^{1/2}, k_{12}]^2 = \begin{pmatrix} -164 & 0 & 0 & -164 \\ 0 & -200 & 160 & 0 \\ 0 & 160 & -128 & 0 \\ -164 & 0 & 0 & -164 \end{pmatrix}$$

and therefore

$$S(\rho_{12}, k_{12}) = \frac{1}{2} \text{Tr}[\rho_{12}^{1/2}, k_{12}]^2 = -328.$$

We calculate the partial traces

$$\rho_1 = \rho_2 = \begin{pmatrix} 13 & 10 \\ 10 & 13 \end{pmatrix}$$

with eigenvalues (3, 23) and positive definite square roots

$$\rho_1^{1/2} = \rho_2^{1/2} = \begin{pmatrix} \frac{\sqrt{3}+\sqrt{23}}{2} & \frac{-\sqrt{3}+\sqrt{23}}{2} \\ \frac{-\sqrt{3}+\sqrt{23}}{2} & \frac{\sqrt{3}+\sqrt{23}}{2} \end{pmatrix}.$$

The commutators

$$[\rho_1^{1/2}, k_1] = -[\rho_2^{1/2}, k_2] = \begin{pmatrix} 0 & -\frac{9}{2}(-\sqrt{3} + \sqrt{23}) \\ \frac{9}{2}(-\sqrt{3} + \sqrt{23}) & 0 \end{pmatrix}$$

have squares

$$[\rho_1^{1/2}, k_1]^2 = [\rho_2^{1/2}, k_2]^2 = \begin{pmatrix} -\frac{81}{4}(-\sqrt{3} + \sqrt{23})^2 & 0 \\ 0 & -\frac{81}{4}(-\sqrt{3} + \sqrt{23})^2 \end{pmatrix}$$

and therefore

$$S(\rho_1, k_1) = S(\rho_2, k_2) = \frac{1}{2} \text{Tr}[\rho_1^{1/2}, k_1]^2 = -\frac{81}{4}(-\sqrt{3} + \sqrt{23})^2.$$

In conclusion we obtain

$$\begin{aligned} S(\rho_1, k_1) + S(\rho_2, k_2) - S(\rho_{12}, k_{12}) &= -\frac{81}{2}(-\sqrt{3} + \sqrt{23})^2 + 328 \\ &= -725 + 81\sqrt{69} \\ &\approx -52.1635, \end{aligned}$$

which contradicts the conjecture of subadditivity of the Wigner-Yanase entropy. We report without giving a proof that also the Wigner-Yanase-Dyson entropies, for $0 < p < 1$, fail to be subadditive.

3. MORE GENERAL ENTROPIES

The failure of the WYD-entropies $S_p(\rho, k)$ to be subadditive is not related to the term ρ^p as one might expect. In a recent paper⁽³⁾ we introduced, for each regular monotone metric with Morozova-Chentsov function c , a so called metric adjusted skew information $I^c(\rho, k)$ which is a generalization of the Wigner-Yanase-Dyson skew information. The extreme points in the convex set of Morozova-Chentsov functions are given by the functions

$$c_\lambda(x, y) = \frac{1 + \lambda}{2} \left(\frac{1}{x + \lambda y} + \frac{1}{\lambda x + y} \right) \quad x, y > 0, \tag{3}$$

where $\lambda \in [0, 1]$. We introduce for each $\lambda \in (0, 1]$ the λ -entropy

$$E_\lambda(\rho, k) = -I^{c_\lambda}(\rho, k) = -\text{Tr} \rho k^2 + \text{Tr} k f_\lambda(L_\rho, R_\rho) k \quad (4)$$

specified by the function

$$f_\lambda(x, y) = xy \cdot c_\lambda(x, y) \quad x, y > 0.$$

Each λ -entropy is a concave function in ρ , and it is additive with respect to the aggregation of isolated subsystems.⁽³⁾ The λ -entropies are vanishing for commuting operators ρ and k and have therefore no classical counterparts. Furthermore, the Wigner-Yanase-Dyson entropy has the following representation

$$S_p(\rho, k) = \frac{p(1-p)}{2} \int_0^1 E_\lambda(\rho, k) \frac{(1+\lambda)^2}{\lambda} d\mu_p(\lambda) \quad 0 < p < 1, \quad (5)$$

where

$$d\mu_p(\lambda) = \frac{2 \sin p\pi}{\pi p(1-p)} \cdot \frac{\lambda^p + \lambda^{1-p}}{(1+\lambda)^3} d\lambda$$

is a probability measure on $[0, 1]$ such that $\int_0^1 \lambda^{-1} d\mu(\lambda) < \infty$.

Although the λ -entropies are much simpler than the WYD-entropies, we realize from Eq. (5) that at least one of them must fail to be subadditive, and this applies then to all of them because of their similarities.

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